Introduction to Interferometry – Part II –

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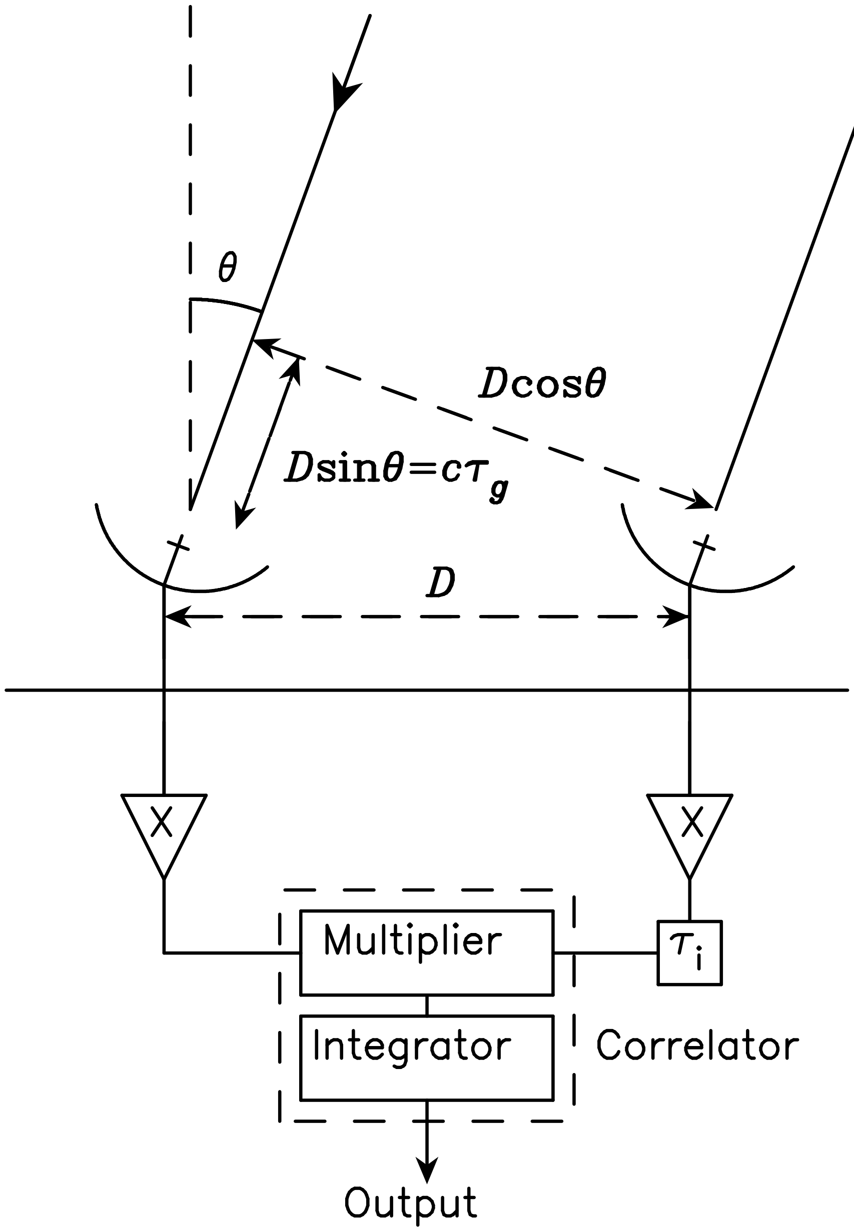
Elementary Interferometer



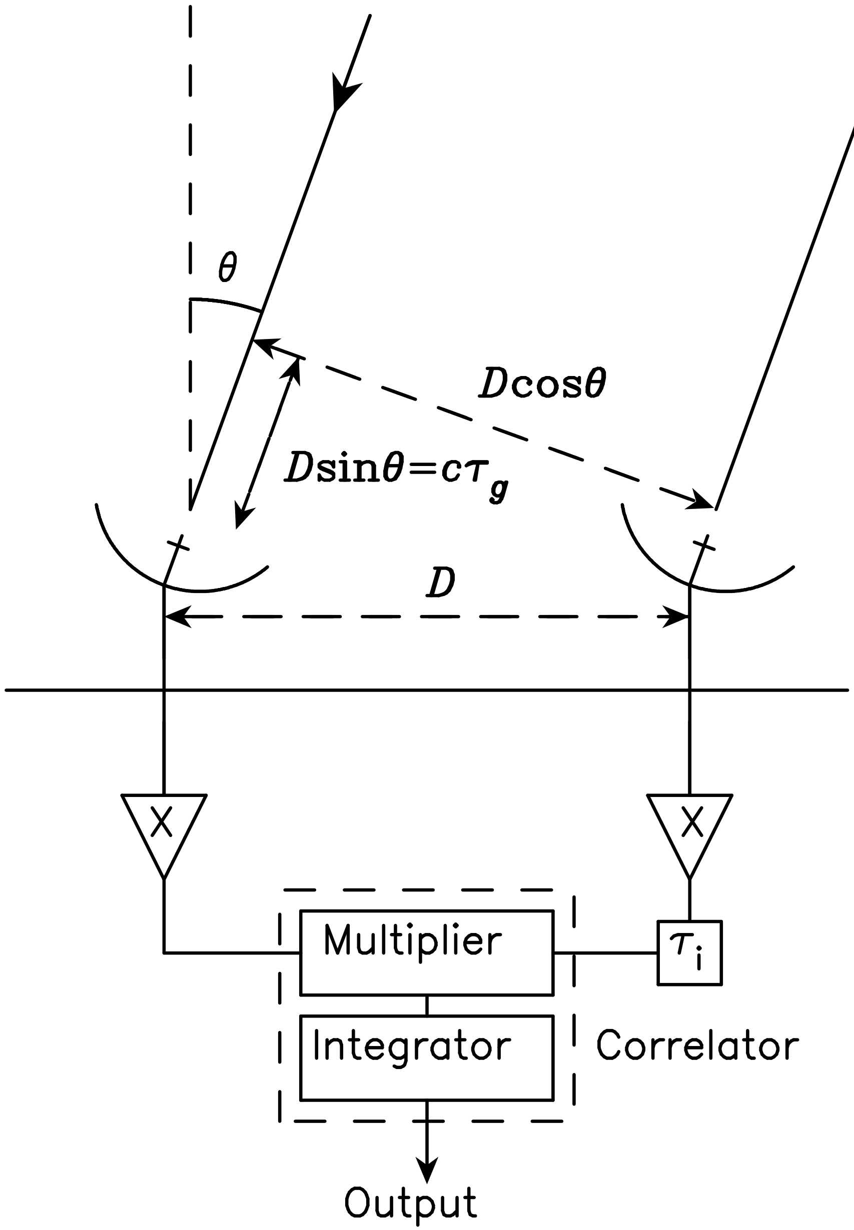
- D: baseline length
- θ: angle of the pointing direction from the zenith, changing with earth rotation.
- τ_g: geometrical delay $= \frac{D}{c} \sin\theta$

The wavefront from the source in direction θ is essentially planar because of great distance traveled, and it reaches the right-hand antenna at a time τ_g before it reaches the left-hand one:

Right: $E\cos(2\pi\nu(t-\tau_g))$ Left: $E\cos(2\pi\nu t)$

The projected length of the baseline on the sky, $D\cos\theta$, changes as the earth rotates.

Elementary Interferometer



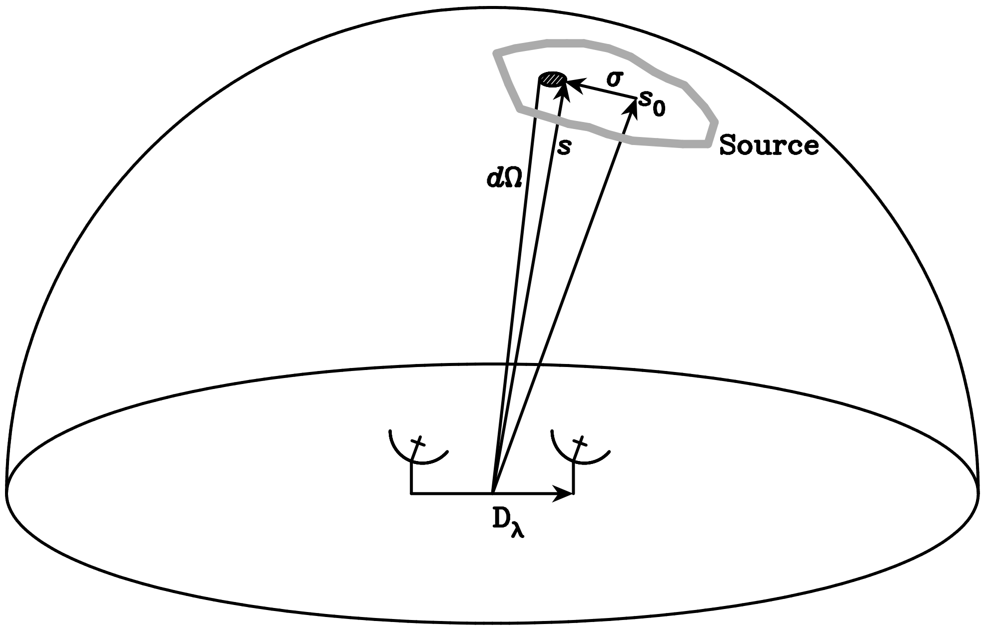
- τ_g: Geometrical time delay
- τ_i: Instrumental time delay
- X: Bandpass amplifiers.
- Correlator: Multiplier + Integrator
- Output: Fringe Visibility

Multiplier:
Multiply the signals from the two antennas.
$\Rightarrow$ Signals are combined by pairs!

Integrator:
Time Averaging Circuit, e.g., 30 sec integration time for each scan in a SMA observation.

Radio Interferometer in reality

Consider an interferometer tracking a source with the pointing center at $\boldsymbol{s}_0$.



Here $\boldsymbol{s}_0$ and $\boldsymbol{s}$ are unit vectors with $\boldsymbol{s} = \boldsymbol{s}_0 + \boldsymbol{\sigma}$ and $\boldsymbol{D}_\lambda = \boldsymbol{D}/\lambda$. The geometrical delay is

$$\tau_g = \frac{\boldsymbol{D} \cdot \boldsymbol{s}}{c} = \frac{\boldsymbol{D}_\lambda \cdot \boldsymbol{s}}{\nu} = \frac{\boldsymbol{D}_\lambda \cdot (\boldsymbol{s}_0 + \boldsymbol{\sigma})}{\nu}$$

where $\boldsymbol{D}_\lambda \cdot \boldsymbol{s}$ is the baseline length projected onto the $\boldsymbol{s}$ direction and $\nu = c/\lambda$.

Complex Visibility

The output from the interferometer is (see last lecture)

$$R(\boldsymbol{D}_\lambda, \boldsymbol{s}_0) = \triangle\nu \int_{4\pi} A(\boldsymbol{\sigma}) B(\boldsymbol{\sigma}) \cos[2\pi\nu(\tau_g - \tau_i)] d\Omega$$

Here, A is the primary beam (i.e., the field of view) of a single antenna, B is the source brightness distribution, and cosine term is the output from the correlator. With

$$\tau_g = \frac{\boldsymbol{D}_\lambda \cdot (\boldsymbol{s}_0 + \boldsymbol{\sigma})}{\nu}$$

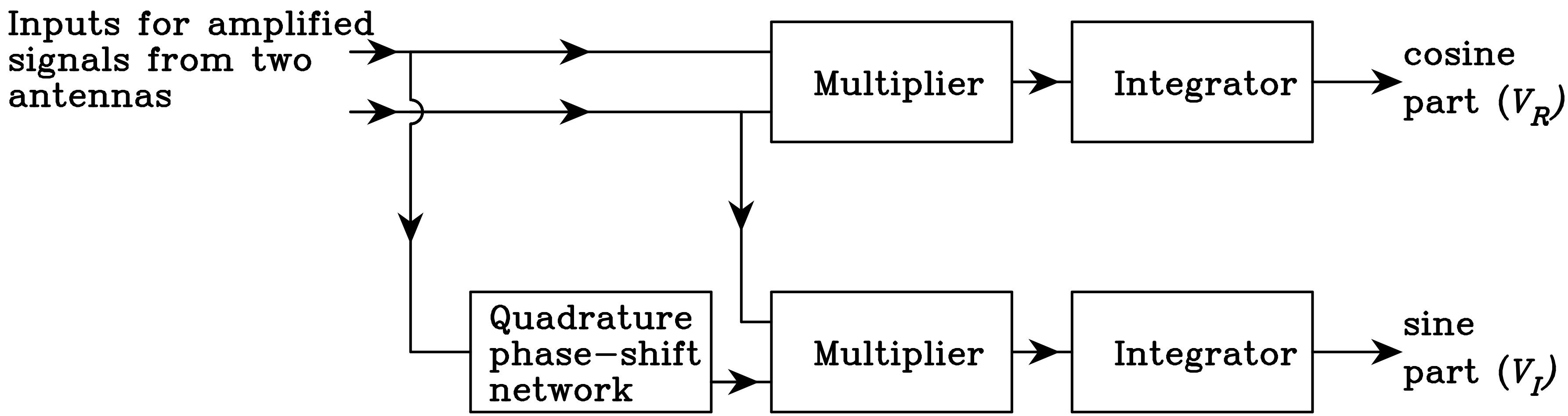
then

$$\begin{aligned} R(\boldsymbol{D}_\lambda, \boldsymbol{s}_0) &= \triangle\nu \int_{4\pi} A(\boldsymbol{\sigma}) B(\boldsymbol{\sigma}) \cos[2\pi \boldsymbol{D}_\lambda \cdot (\boldsymbol{s}_0 + \boldsymbol{\sigma}) - 2\pi\nu\tau_i] d\Omega \\ &= \triangle\nu \int_{4\pi} A(\boldsymbol{\sigma}) B(\boldsymbol{\sigma}) \cos[(2\pi \boldsymbol{D}_\lambda \cdot \boldsymbol{s}_0 - 2\pi\nu\tau_i) + 2\pi \boldsymbol{D}_\lambda \cdot \boldsymbol{\sigma}] d\Omega \\ &= \triangle\nu \cos(2\pi \boldsymbol{D}_\lambda \cdot \boldsymbol{s}_0 - 2\pi\nu\tau_i) \int_{4\pi} A(\boldsymbol{\sigma}) B(\boldsymbol{\sigma}) \cos(2\pi \boldsymbol{D}_\lambda \cdot \boldsymbol{\sigma}) d\Omega \\ &\quad - \triangle\nu \sin(2\pi \boldsymbol{D}_\lambda \cdot \boldsymbol{s}_0 - 2\pi\nu\tau_i) \int_{4\pi} A(\boldsymbol{\sigma}) B(\boldsymbol{\sigma}) \sin(2\pi \boldsymbol{D}_\lambda \cdot \boldsymbol{\sigma}) d\Omega \end{aligned}$$

Complex Visibility, with cosine and sine components? Do we need the two components?

Complex Correlator

To retrieve the sine term, we can add additional correlator with additional time delay of $\frac{1}{4\nu}$ (corresponding to a phase delay of $\frac{\pi}{2}$) so that $\tau_i = \tau_g + \frac{1}{4\nu}$ before the multiplier.



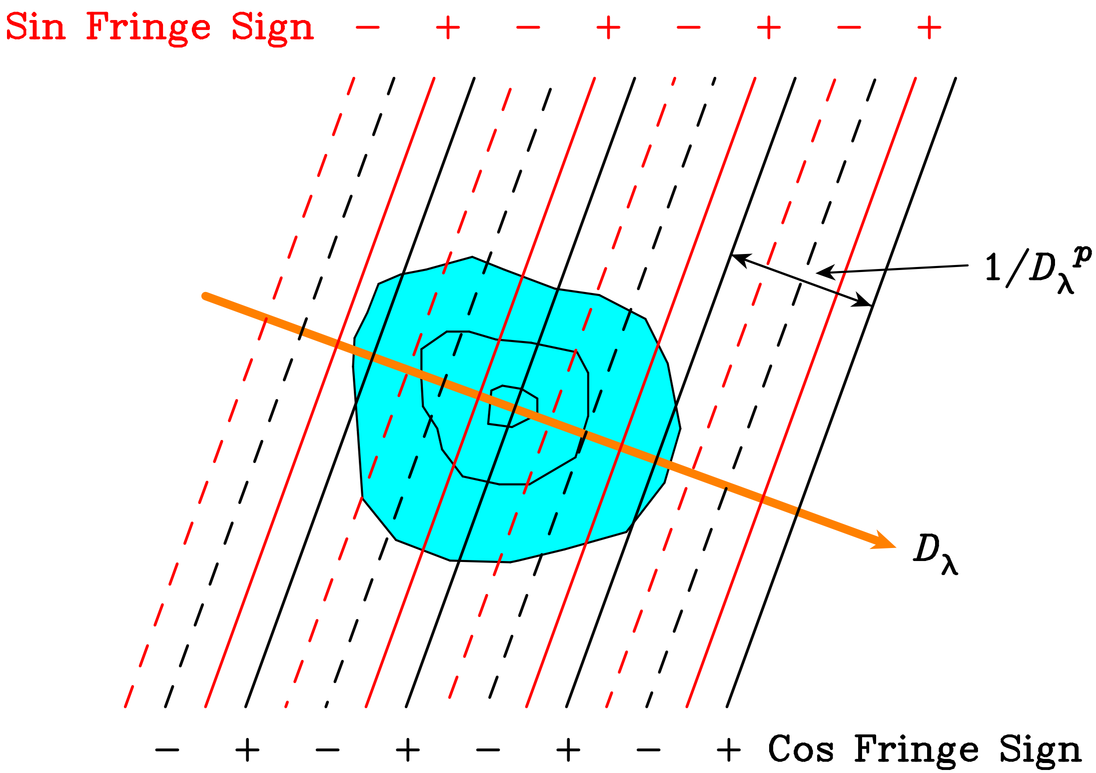
This complex correlator, with two correlators, can measure both components. Here, Quadrature means $\frac{\lambda}{4}$ or $\frac{\pi}{2}$. As a result,

$$\begin{aligned} R(\boldsymbol{D}_\lambda, \boldsymbol{s}_0) &\propto \int_{4\pi} A(\boldsymbol{\sigma})B(\boldsymbol{\sigma})\cos(2\pi \boldsymbol{D}_\lambda \cdot \boldsymbol{\sigma})d\Omega - i\int_{4\pi} A(\boldsymbol{\sigma})B(\boldsymbol{\sigma})\sin(2\pi \boldsymbol{D}_\lambda \cdot \boldsymbol{\sigma})d\Omega \\ &= \int_{4\pi} A(\boldsymbol{\sigma})B(\boldsymbol{\sigma})e^{-i2\pi \boldsymbol{D}_\lambda \cdot \boldsymbol{\sigma}} d\Omega \equiv \mathcal{V}(\boldsymbol{D}_\lambda) \end{aligned} \quad (1)$$

==> the output is a complex visibility $\mathcal{V}$, as a Fourier Transform of $A(\boldsymbol{\sigma})B(\boldsymbol{\sigma})$, with cosine and sine components being the real and imaginary components.

Radio Interferometer

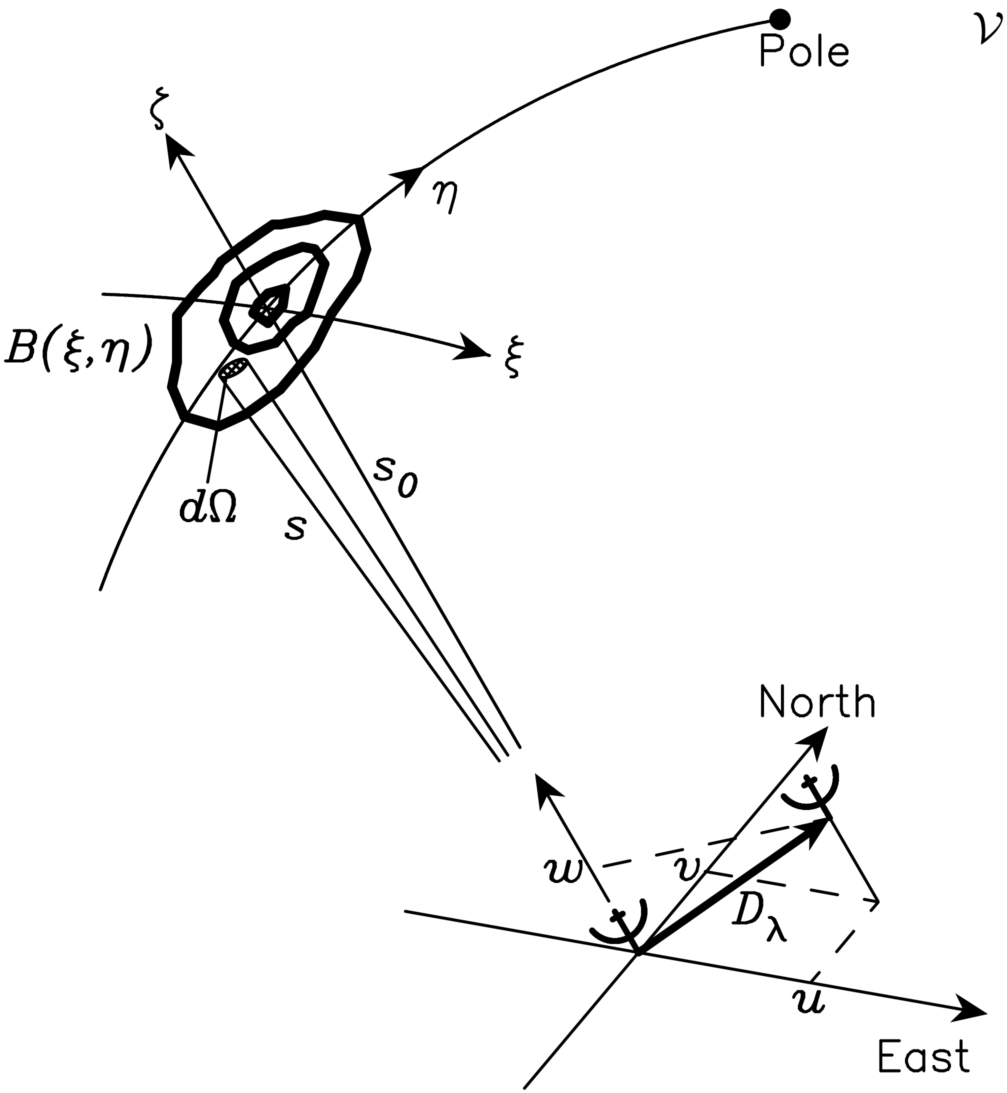
The correlator can be thought of "casting" two sinusoidal fringe patterns of angular scale $1/\boldsymbol{D}_\lambda^p$ radians, onto the sky. $\boldsymbol{D}_\lambda^p$: projected baseline. The correlator multiplies the source brightness by these wave patterns, and integrates the result over the primary beam.



Fringe separation (angular scale) is $1/\boldsymbol{D}_\lambda^p$ radian, with $\boldsymbol{D}_\lambda^p = \boldsymbol{D}_\lambda \cos\theta_0$

Choosing appropriate Coor. System

To solve the visibility function, choose an appropriate Coordinate System (u, v, w) v.s. (ξ, η, ζ)



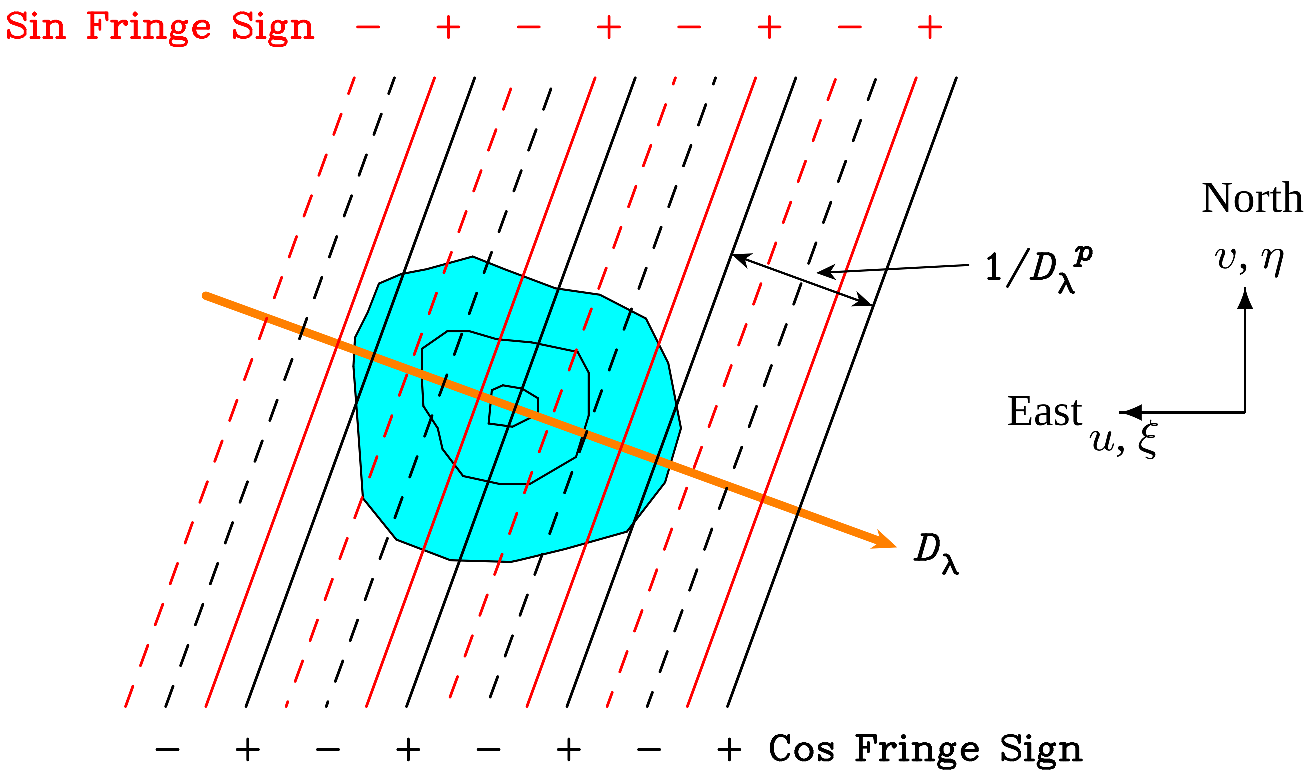
$$\mathcal{V}(\boldsymbol{D}_\lambda) = \int_{4\pi} A(\boldsymbol{\sigma}) B(\boldsymbol{\sigma}) e^{-i2\pi \boldsymbol{D}_\lambda \cdot \boldsymbol{\sigma}} d\Omega$$

$$\begin{aligned}
\boldsymbol{D}_\lambda &= (u, v, w) \\
\boldsymbol{s} &= (\xi, \eta, \zeta) \\
\boldsymbol{s}_0 &= (0, 0, 1) \\
\boldsymbol{\sigma} &= \boldsymbol{s} - \boldsymbol{s}_0 = (\xi, \eta, \zeta - 1) \\
d\Omega &= \frac{d\xi d\eta}{\zeta}
\end{aligned}$$

Here $w \rightarrow \boldsymbol{s}_o$, i.e. pointing center.
and (u, v, w) rotates as the earth rotates.
ξ, η, ζ: direction cosines of $\boldsymbol{s}$ on u, v and w.
$\xi^2 + \eta^2 + \zeta^2 = 1$, $\zeta = \sqrt{1 - \xi^2 - \eta^2}$

Radio Interferometer

With the chosen coordinate system, the plane of the sky is then



Visibility function from 3D to 2D

With the coordinate system, we have

$$\mathcal{V}(u,v,w) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(\xi,\eta)B(\xi,\eta)\exp\{-i2\pi[u\xi + v\eta + w(\sqrt{1-\xi^2-\eta^2}-1)]\} \times \frac{d\xi d\eta}{\sqrt{1-\xi^2-\eta^2}}$$

Now since A drops rapidly when $\xi^2+\eta^2 > l^2$, where l is the full width of the primary beam of the antenna and $l^2 \ll 1$, we only need to consider small ξ and η. In that case,

$$w(\sqrt{1-\xi^2-\eta^2}-1) \simeq -\frac{1}{2}(\xi^2+\eta^2)w$$

i.e, higher order of ξ and η can be neglected. Then

$$\mathcal{V}(u,v,w) \simeq \mathcal{V}(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{A(\xi,\eta)B(\xi,\eta)}{\sqrt{1-\xi^2-\eta^2}} e^{-i2\pi(u\xi+v\eta)} d\xi d\eta$$

Thus, $\mathcal{V}$ is approximately independent of w, and can be considered to be on the flat uv plane. This is because the field of view is so small that each small part of a sphere can be considered as a flat plane.

Mapping: Inverse Fourier Trans.

How to retrieve the source brightness distribution B? Take the the inverse Fourier transform, we have

$$\frac{A(\xi,\eta)B(\xi,\eta)}{\sqrt{1-\xi^2-\eta^2}} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathcal{V}(u,v)e^{i2\pi(u\xi+v\eta)}dudv$$

and then

$$B(\xi,\eta) = \frac{\sqrt{1-\xi^2-\eta^2}}{A(\xi,\eta)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathcal{V}(u,v)e^{i2\pi(u\xi+v\eta)}dudv \propto \mathcal{F}^{-1}[\mathcal{V}]$$

However, the uv plane is not fully sampled. The coverage in the uv plane is sampled by available baselines. Thus, introducing a sampling function $S(u,v)$, we have a "dirty image":

$$B^D(\xi,\eta) = \frac{\sqrt{1-\xi^2-\eta^2}}{A(\xi,\eta)} \sum_k \mathcal{V}(u_k,v_k)S(u_k,v_k)e^{i2\pi(u_k\xi+v_k\eta)}dudv \propto \mathcal{F}^{-1}[\mathcal{V}S]$$

with the sampling function

$$S(u_k,v_k) = \delta(u-u_k, v-v_k)$$

Dirty image and Deconvolution

So what we have is the "dirty image" obtained via an inverse Fourier transform:

$$\begin{aligned} B^D(\xi,\eta) &\propto \mathcal{F}^{-1}[\mathcal{V}S] \\ &\propto \mathcal{F}^{-1}[\mathcal{V}] \otimes \mathcal{F}^{-1}[S] \\ &= B(\xi,\eta) \otimes b(\xi,\eta) \end{aligned}$$

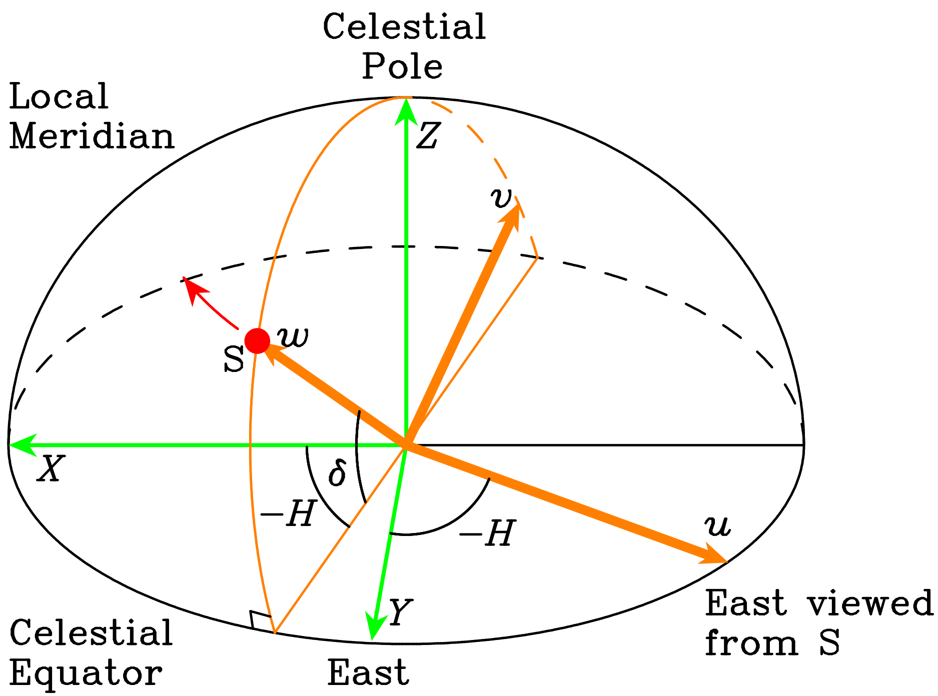
where b is the dirty beam [point spread function (PSF)] given by

$$b(\xi,\eta) \equiv \mathcal{F}^{-1}[S] = \sum_k S(u_k,v_k) e^{i2\pi(u_k\xi+v_k\eta)} du dv$$

Thus, what we really have is the image brightness convolved with the dirty beam. To retrieve the image brightness, we need deconvolution by this dirty beam.

The main lobe of this dirty beam (PSF) can be fitted by a Gaussian beam and is called the synthesized beam. This synthesized beam determines the angular resolution and its size depends on $\lambda/D_{\rm max}$.

Antenna Spacing Coordinate



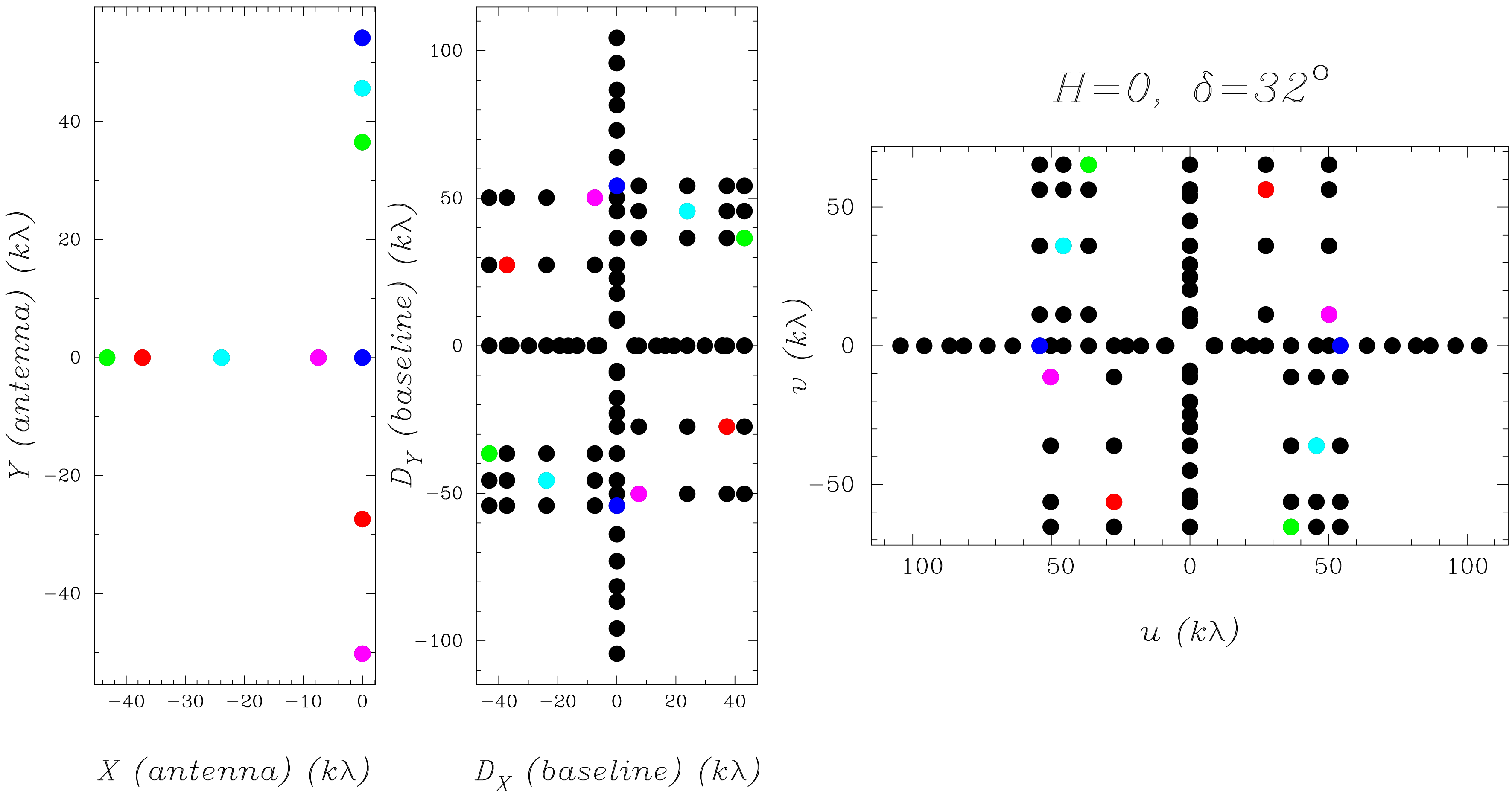
Here (X, Y, Z) is a right-handed Cartesian coordinate system used to specify the relative positions of the antennas in the array. X and Y are measured in a plane parallel to the earth's equator, X in the meridian plane, Y toward the east, and Z is measured toward the north pole. Here, celestial pole is just the earth's pole extended into space, celestial equator is the earth's equator. Source S rises in the east, passes through the local meridian and then sets in the west. Note that u is in the XY plane. H changes as the earth rotates, producing different (u, v, w). When the source at Zenith, $H = 0$, then $u = D_Y$ and $v = -D_X \sin\delta + D_Z \cos\delta$, with (D_X, D_Y, D_Z) being baseline in unit of λ.

Example: BIMA

Size: 6 meter; Number of antennas: $n = 10$; Number of baselines: $n(n-1)/2 = 45$.

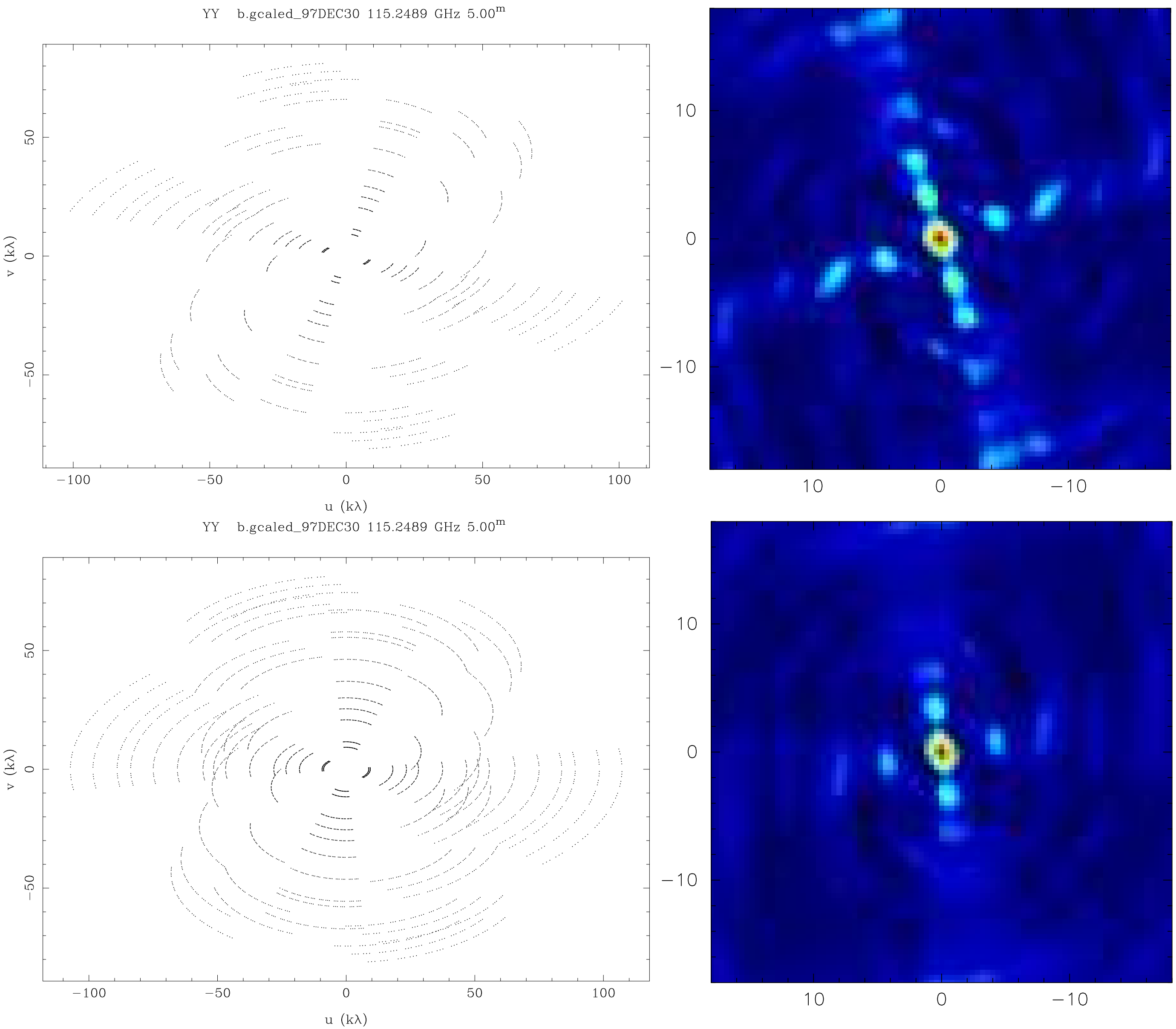
Example: BIMA Zenith snapshot

Number of antennas: $n = 10$, number of (unique) baselines: $n(n-1)/2 = 45$.



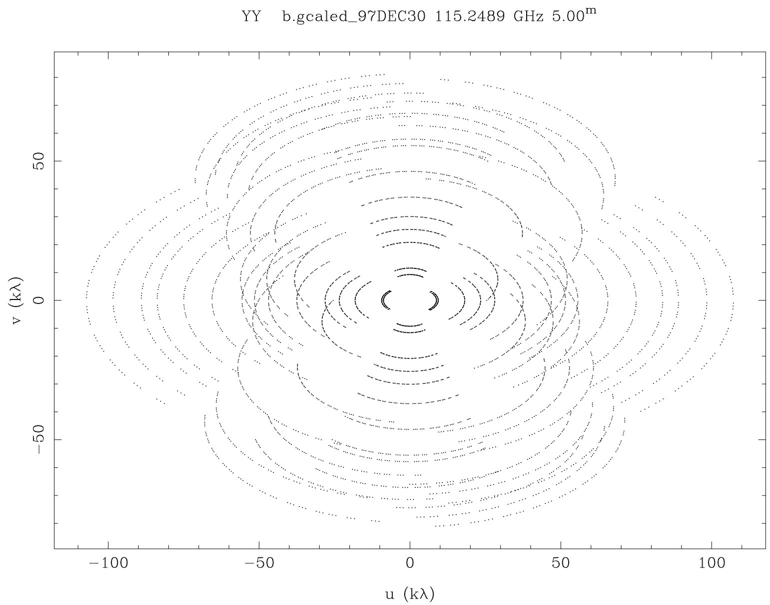
Each unique baseline supplies simultaneously measurements on two uv points. When the source at Zenith, $H = 0$, then $u = D_Y$ and $v = -D_X \sin\delta + D_Z \cos\delta$.

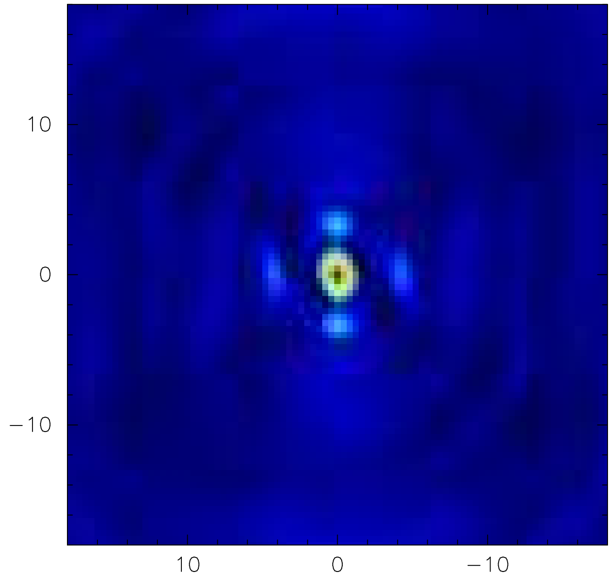
UV coverage and Dirty Beam

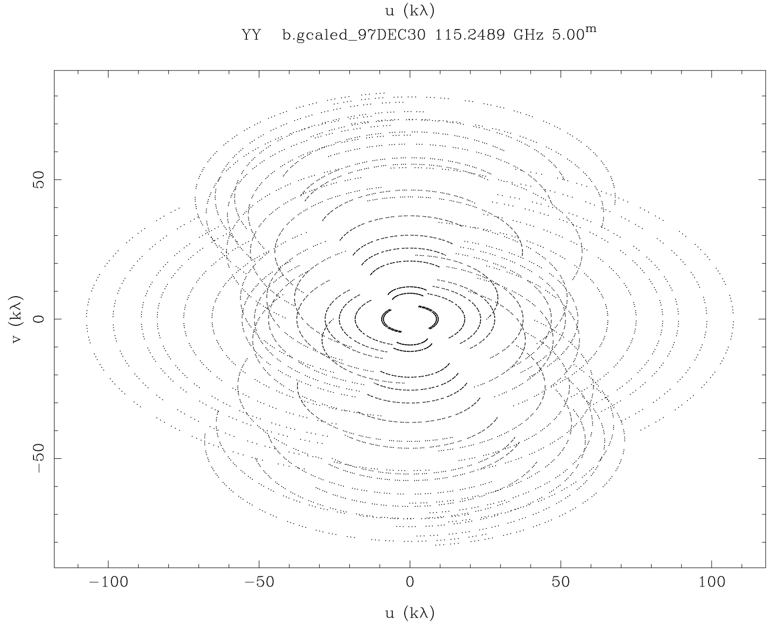


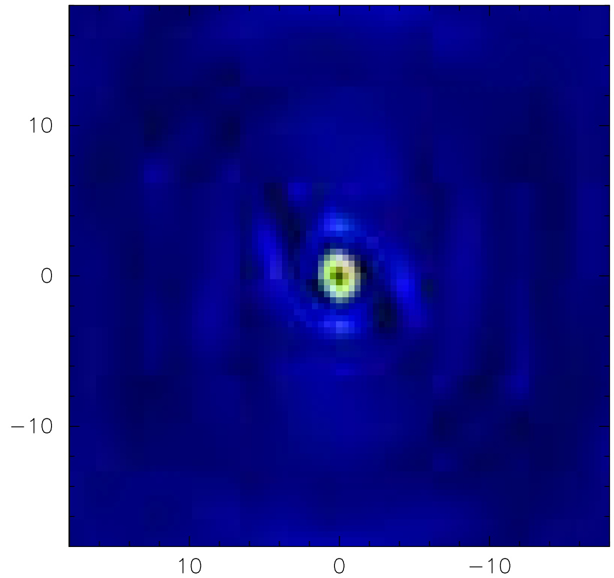
Traditionally, u increases to the right, while ξ increases to the left!

Dirty beam - continue



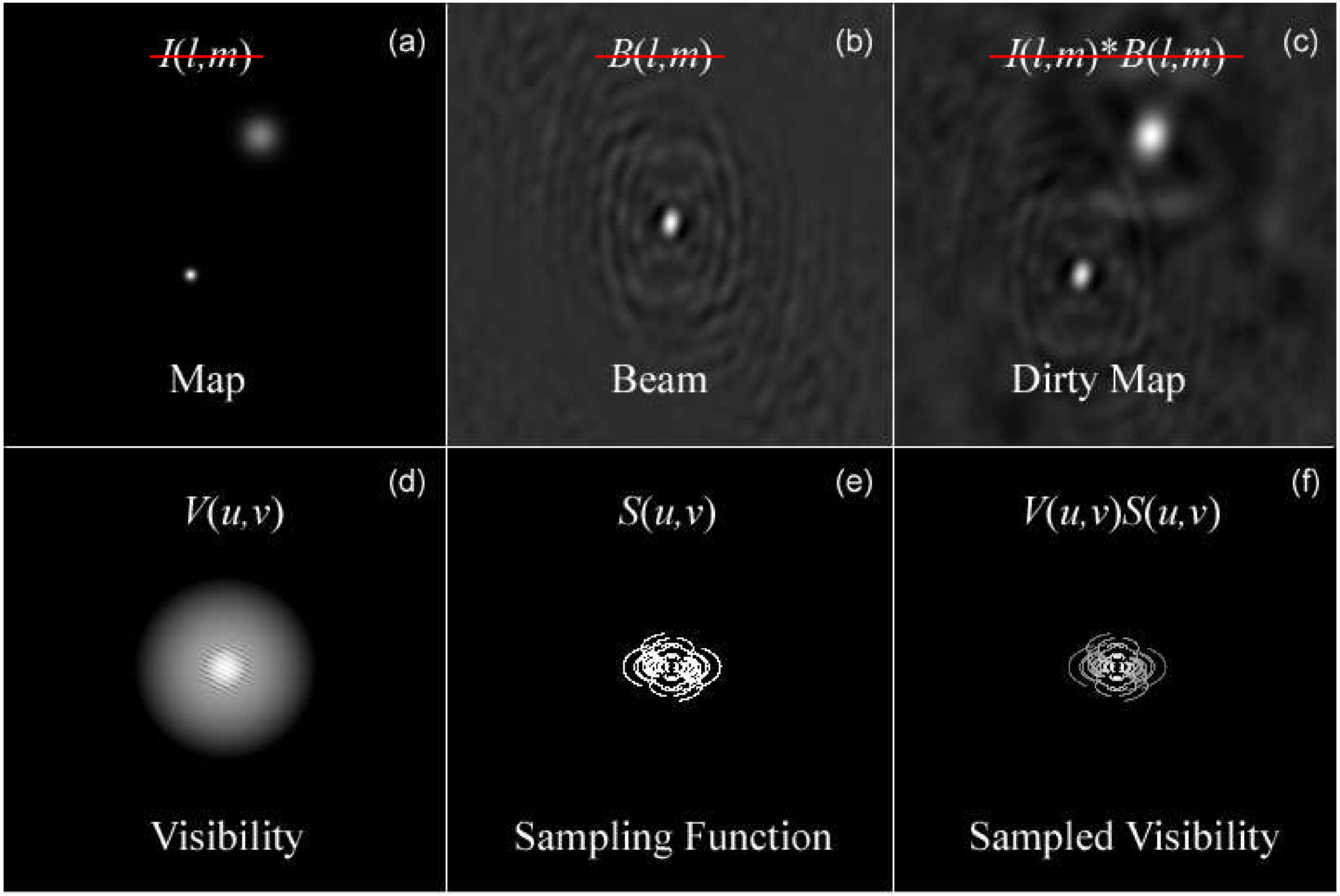






Example: Two Gaussians

$B(\xi,\eta)$ $\quad b(\xi,\eta)$ $\quad B(\xi,\eta)\otimes b(\xi,\eta)$



Interferometer Sensitivity

For one single antenna, the RMS fluctuation (noise) in antenna temperature is (see last lecture of Prof Chin)

$$\triangle T_A = \frac{M T_{\rm sys}}{\sqrt{t \triangle \nu}}$$

where M is a factor of order unity used to account for extra noise from analog to digital conversions, digital clipping etc. Thus, the fluctuation (noise) in flux density is:

$$\triangle S_\nu = \Gamma^{-1} \triangle T_A = \frac{2k}{\eta A} \frac{M T_{\rm sys}}{\sqrt{t \triangle \nu}}$$

where $\Gamma = \frac{\eta A}{2k}$ (usually with a unit of K Jy^{-1}) is the system sensitivity for one single antenna with the aperture area A and the aperture efficiency $\eta < 1$.

For a two-element system (i.e. interferometer), the fluctuation (noise) becomes:

$$\triangle S_\nu = \frac{2k}{\eta A} \frac{M T_{\rm sys}}{\sqrt{2 t \triangle \nu}}$$

The extra factor of $\sqrt{2}$ arises from the use of 2 antennas, and the fact that the correlation of two noisy signals (samples) leads to an increase in the noise by a factor of $\sqrt{2}$.

Interferometer Sensitivity

For an array of n identical telescopes, there are $N = n(n-1)/2$ simultaneous pair-wise correlations. Then the noise in flux density becomes

$$\triangle S_\nu = \frac{2k}{\eta A}\frac{MT_{\rm sys}}{\sqrt{2Nt\triangle\nu}} = \frac{2k}{\eta A}\frac{MT_{\rm sys}}{\sqrt{n(n-1)t\triangle\nu}}$$

Thus, the noise is inversely proportional to $\sqrt{n(n-1)}A$, which approaches the total collecting area of the array for large n. Inserting the numbers, we have

$$\triangle S_\nu = 1.45\,\frac{MT_{\rm sys}}{\eta A\sqrt{n(n-1)t\triangle\nu}} \quad \text{(Jy)}$$

Here t in hr, $\triangle\nu$ in kHz, and A in m^2. Here Jy $= 10^{-23}$ erg s^{-1} cm^{-2} Hz^{-1}. Let's check the performance of the SMA at 1.3 mm (for 230 GHz) in the continuum measurement and let M=1.

For SMA, $n = 8$, $A = 36$ m^2, $\eta \sim 0.7$, 4 hr integration on source with $T_{\rm sys} = 250$ K at transit and $\triangle\nu = 4$ GHz, $\triangle S_\nu = 0.48$ mJy. In reality, $T_{\rm sys}$ increases with decreasing elevation. So the noise could actually be a factor of 2 higher.

How about ALMA? In ES, it has 16 antennas each with a diameter of 12 m. How much faster, assuming the same $\triangle\nu$ and η, but with 25% better in $T_{\rm sys}$ at higher altitude at 5000 meter?

ALMA Spec in ES

- Array: 16 12-m antennas
- Receiver Bands: 115 GHz, 230 GHz, 345 GHz, 690 GHz
- Corresponding Typical lines: CO 1-0, CO 2-1, CO 3-2, CO 6-5
- Primary beam (field of view): 45", 22", 15", 7"
- baseline length D: at least 250 m (may reach 1 km).
- Synthesized Beam (Angular Resolution): at least 2", 1", 0.7", 0.35" (may be 4 times higher)
- Continuum Sensitivity (8 GHz bandwidth in 1 min) (mJy): 0.3, 0.4, 0.9, 3.1

Note that at 230 GHz, SMA with 4GHz bandwidth will take about 4 hr to achieve that sensitivity.